

Groups with f -generics in NTP_2 and PRC fields

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Abstract

We study groups with f -generic types definable in bounded PRC fields. Along the way, we generalize part of the basic theory of definably amenable NIP groups to NTP_2 theories and prove variations on Hrushovski's stabilizer theorem.

1 Introduction

A field is PRC if every absolutely irreducible variety which has zeros in every real closed extension has a zero in the field. Hence PRC fields generalize both the notions of real closed fields and of pseudo algebraically closed fields (PAC). It was shown in [Mon16] that bounded PRC fields are NTP_2 , a notion which generalizes the more studied concepts of dependent theories and simple theories. Since bounded PAC fields have been a very inspirational example of a simple unstable field, and real closed fields are one of the most well studied dependent fields, bounded PRC fields appear to be examples of NTP_2 fields, the study of which can be very telling about which properties one can and cannot expect of an NTP_2 theory.

In this paper we try to understand definable groups in a bounded PRC field, assuming in addition existence of f -generic types (a slightly weaker assumption than definable amenability). We prove that such a group is isogeneous with a finite index subgroup of a quantifier-free definable groups (Theorem 6.3). In fact, that latter group admits a definable covering by multicells on which the group operation is algebraic. This generalizes similar

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results proved in [HP94] by Hrushovski and Pillay for (not necessarily f-generic) groups definable in both pseudofinite fields and real closed fields. Our theorem applies in particular to all solvable groups.

In order to prove this result we need to develop various tools. In Section 2 we prove two new versions of Hrushovski's Stabilizer Theorem from [Hru12]. In particular, we manage to give a slightly simpler proof at the cost of losing some optimality in the hypothesis. In Section 3 we prove some results about groups definable in an NTP_2 theory admitting f-generic types. We generalize some basic statements proved in [CS16] for definably amenable NIP groups. In Section 4 we recall some results on PRC fields and prove that the expansion of a bounded PRC field obtained by adding all quantifier-free externally definable sets has elimination of quantifiers.

The sketch of the proof of the main theorem is as follows: After an initial reduction to groups of finite index, we use the same ideas of the first sections of [HP94] to show that given a group G with f-generics definable in a bounded PRC field, there is an algebraic group H and a (relatively) definable isomorphism between type definable subgroups G_M^{00} of G and K of H . The isomorphism is achieved from the maximum type definable over M subgroup G_M^{00} of G , a fact uses very strongly that G has f-generics and the Stabilizer Theorem. In Section 5 we show that for any such K (a type definable subgroup of and algebraic group), if \overline{K} denotes the topological closure of K , then \overline{K}/K is profinite. The proof then continues adapting the proofs in [HP94] for the pseudofinite case and for the real closed case to complete the proof of the bounded PRC case.

2 Stabilizer theorems

Let M be a model and let G be an M -definable group. Let μ be an M -invariant ideal of subsets of G which is invariant by left translations by elements of G . We say that a type $p(x)$ in G is μ -wide if it is not contained in a set $D \in \mu$. A key concept we will need is Hrushovski's definition of an S1 ideal.

Definition 2.1. An A -invariant ideal μ has the S1 property if whenever $(a_j)_{j \in \omega}$ is an A -indiscernible sequence and $\phi(x, y)$ is a formula, then if $\phi(x, a_i) \wedge \phi(x, a_j)$ is in μ for some/all $i \neq j$, then $\phi(x, a_i)$ is in μ for some/any i .

We will say that the ideal μ is S1 on the A -definable set X if X is not in μ and the property above holds for formulas $\phi(x, a_i)$ included in X . Finally,

we say that μ is S1 on a partial type $\pi(x)$ if $\pi(x)$ is μ -wide and included in a definable set in which μ is S1.

The following results all appear in [Hru12].

Fact 2.2. *Let p be a type and assume that μ has the S1 property. Then for any type q the relation*

$$R(a, b) \iff p(x)a^{-1} \cap q(x)b^{-1} \text{ is } \mu\text{-wide,}$$

where we identify a type with its realizations in the monster model, is a stable relation.

Fact 2.3. *Let μ be an M -invariant ideal which is S1 on some set X . Then for any type $p(x)$ whose realizations are contained in X , if $p(x)$ is μ -wide then $p(x)$ does not fork over M .*

Finally, the following is Lemma 2.3 in [Hru12].

Fact 2.4. *Let p, q be complete types over a model M and let $R(x, y)$ be a stable M -invariant relation in the realizations of $p(x) \times q(y)$. Then the truth value of $R(a, b)$ is constant for all $a \models p(x)$ and $b \models q(y)$ as long as either $\text{tp}(a/Mb)$ or $\text{tp}(b/Ma)$ does not fork over M .*

We consider a second ideal λ of subsets of G with the property that μ is S1 on any set in λ . This ideal will only be truly used in Theorem 2.13; everywhere else, one may take λ to be the ideal of all definable sets on which μ is S1. We assume that λ is also invariant under left translations by elements of G . A type which is not λ -wide will be called *medium*, and we will refer to μ -wide types by “wide”. Note that if p is medium, then μ is S1 on p , and if $a \models p$ and $\text{tp}(a/Mb)$ is wide, then $\text{tp}(a/Mb)$ does not fork over M .

If q and r are wide types, then we define $St(q, r) := \{g : gp \cap r \text{ is wide}\}$. If p is wide, we will refer to $St(p, p)$ by $St(p)$ and $St_r(p) = \{g : pg \cap p \text{ is wide}\}$. Hence $g \in St(p)$ if and only if there is some $a \models p$, $\text{tp}(a/Mg)$ wide and $ga \models p$ (then also $\text{tp}(ga/Mg)$ is wide by G -invariance of μ). Observe that $St(p)$ is stable under inversion. Finally, $Stab(p)$ is the subgroup generated by $St(p)$.

If p and q are two types, we let $p \times_{nf} q = \{(a, b) : a \models p, b \models q, \text{tp}(b/Ma) \text{ does not fork over } M\}$.

We recall one version of Hrushovski’s stabilizer theorem from [Hru12].

Fact 2.5 ([Hru12]). *Let μ be an M -invariant ideal on G stable under left and right multiplication. Let $X \subseteq G$ be a symmetric M -definable set such that μ is S1 on X^3 . Let q be a wide type over M concentrating on X . Assume*

(F) *There are $a, b \models q$ such that $\text{tp}(a/Mb)$ and $\text{tp}(b/Ma)$ are both non-forking over M .*

Then there is a wide type-definable subgroup S of G . We have $S = (q^{-1}q)^2$ and $qq^{-1}q$ is a coset of S . Moreover S is normal in the group generated by X and $S \setminus (q^{-1}q)$ is included in a union of non-wide M -definable sets.

We will not actually use this theorem, but some modified versions of it, which we prove in this section. Theorem 2.10 below is very close to Fact 2.5. The proof is of course very much inspired, at times literally copied, from that of Hrushovski. The main difference is that we assume the ideal to be S1 on up to four products of the type and its inverse (instead of three), and this allows us to simplify slightly the arguments. Furthermore, we drop assumption (F) and under assumption (B1), we forgo right-invariance.

The proof in [Hru12] operates by acting on the right on q , we decide to act on the left, which explains some differences in the statements.

We will need a stronger version of Fact 2.2, where we restrict the requirement that μ has the S1 property in all sets.

Lemma 2.6. *Let p, q be medium, then the relation $R(g, h)$ defined as “ $gp \cap hq$ is wide” is a stable relation.*

Proof. Note that by invariance of μ , every translate of p and q is medium. Let $(g_i h_i : i \in \mathbb{Z})$ be an indiscernible sequence and assume that $R(g_i, h_j)$ holds if and only if $i \leq j$.

Case 1: $g_0 p \cap g_1 p \cap h_2 q$ is wide.

We then have that for all $i > 0$, $g_0 p \cap g_i p \cap h_{i+1} q$ is wide by indiscernibility. Also for $i < j$, we have $(g_i p \cap h_{i+1} q) \cap (g_{i+2} p \cap h_{i+3} q)$ is not wide as already $h_{i+1} q \cap g_{i+2} p$ is not wide. Therefore the sequence $(g_0 p \cap (g_{2i} p \cap h_{2i+1} q) : i > 1)$ contradicts the S1 property inside $g_0 p$.

Case 2: $g_0 p \cap g_1 p \cap h_2 q$ is not wide.

We know that for all $i < 2$, $g_i p \cap h_2 q$ is wide. Hence the sequence $(h_2 q \cap g_i p : i < 2)$ contradicts the S1 property inside $h_2 q$. \square

Lemma 2.7. *Let q, r be medium and wide, and let $p \in St(q, r)$. Take $(a, b) \in p \times_{nf} p$, then $a^{-1}b, b^{-1}a \in St(q)$.*

Proof. Take $(a, b) \in p \times_{nf} p$. Since $St(q)$ is stable under inverses, it suffices to show that $a^{-1}bq \cap q$ is wide, which is equivalent to $bq \cap aq$ is wide. As q is medium, by stability it is enough to prove this for one pair $(a, b) \in p \times_{nf} p$. Take $(a_i : i < \omega)$ an indiscernible sequence in p such that $\text{tp}(a_1/Ma_0)$ is non-forking over M . Then $a_iq \cap r$ is wide for all i , as $p \in St(q, r)$. As r is medium, it follows that $a_0q \cap a_1q \cap r$ is wide. In particular $a_0q \cap a_1q$ is wide, as required. \square

Lemma 2.8. *Let p be wide and medium, and let $q \in St(p)$, take $(a, b) \models q \times_{nf} q$, then $a^{-1}b, b^{-1}a \in St(p)$. If μ is right invariant and if $q \in St_r(p)$, then $ab^{-1}, ba^{-1} \in St_r(p)$.*

Proof. The first part follows from the previous lemma by taking q, r there to be p here. The second part of the statement is proved in the same way by multiplying on the right. \square

We will also show the following.

Lemma 2.9. *Let p and r be medium types, let $(a, b) \models p \times_{nf} p$, and assume that $p^{-1}r$ is medium. Then $ba^{-1} \in St(r)$.*

Proof. We need to show that $a^{-1}r \cap b^{-1}r$ is wide. Let $(a_i)_{i < \omega}$ be an indiscernible sequence of realizations of p such that $\text{tp}(a_n/Ma_{<n})$ is wide for all n (and hence non-forking over M as p is medium). By stability, it is enough to show that $a_0^{-1}r \cap a_1^{-1}r$ is wide. The type-definable sets $(a_i^{-1}r)_{i < \omega}$ are wide and included in $p^{-1}r$ which is medium by hypothesis, so by the S1 property $a_0^{-1}r \cap a_1^{-1}r$ is wide as required. \square

Theorem 2.10. *Let μ be an M -invariant ideal on G stable under left multiplication. Let $p \in S_G(M)$ be wide. Assume either (B1) or (B2), where:*

(B1) *For some symmetric definable set $X \in p$, μ is S1 on X^4 ;*

(B2) *μ is S1 on $(pp^{-1})^2$ and invariant under (left and) right multiplication;*

Then $\text{Stab}(p) = St(p)^2 = (pp^{-1})^2$ is a connected, wide type-definable group on which μ is S1. Furthermore $\text{Stab}(p) \setminus St(p)$ is included in a union of non-wide M -definable sets.

Proof. Here we take for λ the ideal of all definable sets on which μ is S1, so a type is medium if μ is S1 on it. In particular, under either of (B1) or (B2), we have that both p and $p^{-1}p$ are medium.

The proof will proceed by a series of steps. Only in the beginning will there be differences depending on whether (B1) or (B2) is assumed.

Claim 1: Let $(a, b) \in p \times_{nf} p$, then $ba^{-1} \in St(p)$.

Proof: This follows immediately from Lemma 2.9.

Claim 1': If (B1) holds, then for any $(a, b) \in p \times_{nf} p$ we have $a^{-1}b \in St(p)$.

Proof: By symmetry of X , we have that p^2 is medium, so the result follows from Lemma 2.9 with $p = p^{-1}$ and $r = p$.

Take now $(a, b) \in p \times_{nf} p$, $\text{tp}(b/Ma)$ wide. We define $q = \text{tp}(a^{-1}b/M)$ under assumption (B1) and $q = \text{tp}(ba^{-1}/M)$ under assumption (B2). Then in both cases $q \in St(p)$, q is wide (using right-invariance in the (B2) case) and medium. Notice that under either assumption $p^{-1}q$ is medium: under (B1) $p^{-1}q \subseteq X^3$ and under (B2) $p^{-1}q \subseteq p^{-1}pp^{-1}$.

So Lemma 2.9 implies

Claim 2: Let $(a, b) \in p \times_{nf} p$, then $ba^{-1} \in St(q)$.

Claim 3: Let $(b, c) \in Stab(q) \times_{nf} q$, then $bc \in St(p)$.

Proof: As $St(q)$ is stable under inverse, we can write $b = b_1 \cdots b_n$, with each $b_i \in St(q)$. We show the result by induction on n . For $n = 0$, it follows from the fact that $q \in St(p)$.

Assume we know it for $n - 1$ and take $b = b_1 \cdots b_n$. We have to show that $b_n^{-1} \cdots b_1^{-1}p \cap cp$ is wide. As $b_n \in St(q)$, there is $c' \models q$, $\text{tp}(c'/Mb_n)$ wide such that $b_n c' \models q$. We may also assume that $\text{tp}(c'/Mb_0 \dots b_n)$ is wide. Then by translation invariance, $\text{tp}(b_n c'/Mb_0 \dots b_n)$ is wide. By induction, $b_{n-1}^{-1} \cdots b_1^{-1}p \cap b_n c' p$ is wide, then so is $b_n^{-1} \cdots b_1^{-1}p \cap c' p$ and we conclude by stability.

Claim 4: Let $a, b \models p$, then $ab^{-1} \in St(q)^2$.

Proof: Take $c \models p$ such that $\text{tp}(c/Mab)$ is non-forking over M . Write $ab^{-1} = (ac^{-1})(cb^{-1})$. By Claim 2 and the fact that $St(q)$ is closed under inverses, both ac^{-1} and cb^{-1} are in $St(q)$ and the claim follows.

Claim 5: $Stab(p) = Stab(q) = (pp^{-1})^2$ is wide and medium.

Proof: By Claim 3, we have $Stab(q) \subseteq St(p)^2 \subseteq (pp^{-1})^2$. By Claim 4, $pp^{-1} \subseteq Stab(q)$ so also $(pp^{-1})^2 \subseteq Stab(q)$, hence $(pp^{-1})^2 = St(p)^2 = Stab(q)$. Finally, since $Stab(q)$ is a subgroup, we have $Stab(p) = St(p)^2 = Stab(q)$. By hypothesis $(pp^{-1})^2$ is medium, and it is wide since it contains q .

So we just need to prove that $Stab(p)$ has no type-definable over M proper subgroup of bounded index, and that any wide type in $Stab(p)$ lies in $St(p)$.

Let $T \leq Stab(p)$ be a type-definable over M subgroup of bounded index. We have $pp^{-1} \subseteq Stab(p)$, hence for $a \models p$, $p \subseteq Stab(p)a$. So p lies in a right coset S_p of $Stab(p)$. This coset is M -invariant and hence type-definable over M . All right cosets of T in S_p are type-definable over M and as p is a complete type over M , it must lie entirely within one of them. Therefore $pp^{-1} \subseteq T$ and $T = Stab(p)$.

Now, let s be a wide type in $Stab(p) = Stab(q)$. By Claim 3, for any $b \in Stab(q)$ and $c \models q$ with $tp(c/Mb)$ wide, $bp \cap cp$ is wide. By stability, the same holds assuming instead that $tp(b/Mc)$ is wide. Let $c \models q$ and $b \models s$ such that $tp(b/cM)$ is wide. Then by left invariance, $tp(cb/cM)$ is wide. But we also have $cb \in Stab(q)$, hence $cbp \cap cp$ is wide. From which it follows that $bp \cap p$ is wide, so s lies in $St(p)$, as required. \square

Proposition 2.11. *Under assumption (B1), $Stab(p)$ is normal and of bounded index in the group generated by X and X^n is medium for all n .*

Proof. Write $S = Stab(p)$. Let r be a type over M of elements of X . Then the image of r in G/S is bounded. Indeed, assume not, then we can find an indiscernible sequence $(a_i : i < \omega)$ of realizations of r such that the cosets $a_i S$ are pairwise disjoint. Hence so are the types $a_i pp^{-1}$ (as $pp^{-1} \subseteq S$), but this contradicts S1 inside X^3 . As r is a complete type over M it must be included in one left coset of S . Applying the same reasoning to r^{-1} , we see that r is also included in a unique right coset of S . Thus X/S is bounded and if $c, c' \models r$, then $cSc^{-1} = c'Sc'^{-1} =: S^r$ is type-definable over M .

We now claim that p^{-1} has bounded image in G/S^r : for if not, we would have an Mc -indiscernible sequence $(a_i : i < \omega)$ of realizations of p with $a_i^{-1}cSc^{-1}$ pairwise disjoint and again $a_i^{-1}cpp^{-1}$ would be pairwise disjoint contradicting S1 in X^4 . Hence p^{-1} lies entirely within one left coset of S^r and $pp^{-1} \subseteq S^r$. Therefore $S \leq S^r$. We also have $S \leq S^{r^{-1}}$ and then $S = S^r$.

We have shown that S is normalized by X and has bounded index in it. It follows that S has bounded index in any X^n , thus X^n is medium. \square

Proposition 2.12. *If we assume that both conditions (B1) and (B2) (equivalently (B1) and right-invariance) hold, then $pp^{-1}p$ is a coset of $Stab(p)$.*

Proof. Let $c \models p$. By the previous proposition $Stab(p)$ is normal in the group generated by X . Since $pp^{-1} \subseteq Stab(p)$, p lies entirely within one coset of

$Stab(p)$ and hence $pp^{-1}p \subseteq Stab(p)c$. Conversely, take any $a \in Stab(p)c$ and let $b \models p$ such that $tp(b/Ma)$ is wide. Then $ba^{-1} \in Stab(p)$ and $tp(ba^{-1}/M)$ is wide by right-invariance. By Theorem 2.10 any wide type in $Stab(p)$ is in $St(p)$, so $ba^{-1} \in St(p) \subseteq pp^{-1}$. So $a = ab^{-1}b \in pp^{-1}p$. \square

We now wish to relax the hypothesis that μ is S1 on $(pp^{-1})^2$ and assume only that μ is S1 on *generic* products in $p^{-1}p$ (see condition (B) below). We will need however to make extra technical hypothesis (A) and (F).

Theorem 2.13. *Let μ and λ be M -invariant ideals on G as above, stable under left and right multiplication, and such that μ is S1 in any $X \in \lambda$.*

Assume we are given a wide and medium type p in G and the following conditions are satisfied:

(A) *for any types q, r , if for some $(c, d) \models q \times_{nf} r$, $tp(cd/M)$ or $tp(dc/M)$ is medium, then q is medium;*

(B) *for any $(a, b) \in p \times_{nf} p$, $tp(a^{-1}b/M)$ is medium;*

(F) *there are $(a, b) \models p \times_{nf} p$ such that $tp(a/Mb)$ does not fork over M .*

Then $Stab(p) = St(p)^2 = (pp^{-1})^2$ is a connected type-definable, wide and medium group. Also $Stab(p) \setminus St(p)$ is contained in a union of non-wide M -definable sets.

Proof. Condition (A) implies that if q is a medium type, then both $St(q)$ and $St_r(q)$ are medium.

Claim 1: If $(a, b) \in p \times_{nf} p$, then $ba^{-1} \in St(p)$.

Proof: By Lemma 2.9.

Claim 1': If $(a, b) \in p \times_{nf} p$, then $a^{-1}b \in St_r(p)$.

Proof: By Claim 1, we have that if $(a, b) \in p \times_{nf} p$, then $ba^{-1} \in St(p)$, in particular $tp(ba^{-1}/M)$ is medium. We can then repeat the argument of Lemma 2.9 by multiplying on the right to show Claim 1'.

Let μ' be the ideal defined by $\phi(x) \in \mu' \iff \phi(x^{-1}) \in \mu$. Then μ' is M -invariant, invariant under left and right multiplication and is S1 on any inverse of a medium type. We will write St' , $Stab'$ for the stabilizers with respect to μ' .

Let $(a, b) \models p \times p$, $tp(b/Ma)$ wide (hence non-forking over M) and $q = tp(ab^{-1}/M)$. Then q is μ' -wide and is in $St(p)$, as $St(p)$ is closed under

inverses, and thus q and q^{-1} are medium. Also if $(c, d) \models q \times_{nf} q$, then $\text{tp}(c^{-1}d/M) \in St(p)$ by Lemma 2.8. In particular $\text{tp}(c^{-1}d/M)$ is medium.

Claim 2: If $(b, c) \in Stab'(q) \times_{nf} q$, then $bc \in St(p)$.

Proof: As $St'(q)$ is stable under inverse, we can write $b = b_1 \cdots b_n$, with each $b_i \in St'(q)$. We show the result by induction on n . For $n = 0$, it is clear.

Assume we know it for $n - 1$ and take $b = b_1 \cdots b_n$. We have to show that $b_n^{-1} \cdots b_1^{-1}p \cap cp$ is wide. As $b_n \in St'(q)$, there is $c' \models q$, $\text{tp}(c'/Mb_n)$ μ' -wide such that $b_nc' \models q$. We may also assume that $\text{tp}(c'/Mb_1 \dots b_n)$ is μ' -wide. Then by translation invariance, $\text{tp}(b_nc'/Mb_1 \dots b_n)$ is μ' -wide. By induction, $b_{n-1}^{-1} \cdots b_1^{-1}p \cap b_nc'p$ is wide. We conclude by stability.

Claim 3: There is $(a, b) \models p \times_{nf} q$, $\text{tp}(b/Ma)$ μ' -wide, such that $\text{tp}(a^{-1}b/M)$ and its inverse are medium.

Proof: By (F) there is $(c, d) \in p \times_{nf} p$ such that also $\text{tp}(c/Md)$ does not fork over M . Let $r = \text{tp}(d^{-1}c/M)$. Let $a \models p$ and choose b_0 such that $\text{tp}(a, b_0/M) = \text{tp}(d, c/M)$. Then $a^{-1}b_0 \models r$ and $\text{tp}(b_0/Ma)$ does not fork over M . Now choose $b_1 \models p$ such that $\text{tp}(b_1/Mb_0)$ is wide and $\text{tp}(b_0b_1^{-1}/M) = q$. We can furthermore assume that $\text{tp}(b_1/Mab_0)$ is wide. By translation invariance, $\text{tp}(b_0b_1^{-1}/Ma)$ is μ' -wide. Now pick b_2 such that $\text{tp}(b_2/Mab_0b_1)$ is non-forking over M and $\text{tp}(b_1, b_2/M) = \text{tp}(c, d/M)$ so that $\text{tp}(b_1^{-1}b_2/M) = r^{-1}$. By transitivity of non-forking, we have $\text{tp}(b_1^{-1}b_2/Mab_0)$ is non-forking over M . Hence $(a^{-1}b_0, b_1^{-1}b_2) \models r \times_{nf} r^{-1}$.

By Claim 1' and the since $St_r(p)$ is stable under inversion, $r \in St_r(p)$ and by Lemma 2.8, $a^{-1}b_0b_1^{-1}b_2$ is also in $St_r(p)$. It follows that $\text{tp}(a^{-1}b_0b_1^{-1}b_2/M)$ and its inverse are medium. By hypothesis (A), $\text{tp}(a^{-1}b_0b_1^{-1}/M)$ and its inverse are medium.

Claim 4: If $(a, b) \models p \times_{nf} p$, then $ab^{-1} \in St'(q)$.

The proof is similar to that of Lemma 2.9. As there, we can take $(a_i : i < \omega)$ an indiscernible sequence in p with $\text{tp}(a_n/Ma_{<n})$ wide and it is enough to show that $a_0^{-1}q \cap a_1^{-1}q$ is μ' -wide. By Claim 3, there is $b \models q$ with $\text{tp}(b/Ma_{<\omega})$ μ' -wide, $(a_i)_{i < \omega}$ indiscernible over Mb and $r = \text{tp}(a_0^{-1}b/M)$ and its inverse are medium. Also $a_0^{-1}b \in a_0^{-1}q \cap r$. By translation invariance, $\text{tp}(a_0^{-1}b/Ma_0)$ is μ' -wide, hence $a_0^{-1}q \cap r$ is wide. By indiscernibility, $a_i^{-1}q \cap r$ is μ' -wide for all i . As r^{-1} is medium, it follows that $a_0^{-1}q \cap a_1^{-1}q$ is μ' -wide.

Now we can conclude: we have by Claim 2, $Stab'(q) \subseteq St(p)^2 \subseteq (pp^{-1})^2$. Let $a, b \models p$ and choose $c \models p$ such that $\text{tp}(b/Mc)$ and $\text{tp}(c/Mb)$ do not fork over M (using (F)). We can furthermore assume that $\text{tp}(c/Mab)$ does not fork

over M . Then $(a, c) \models p \times_{nf} p$ and $(c, b) \models p \times_{nf} p$ and $ab^{-1} = (ac^{-1})(cb^{-1})$. By Claim 4, both ac^{-1} and cb^{-1} are in $St'(q)$, therefore $ab^{-1} \in Stab'(q)$. We thus have $pp^{-1} \subseteq Stab'(q)$. Therefore $Stab'(q) = St(p)^2 = (pp^{-1})^2$ and as $Stab'(q)$ is a subgroup, $Stab'(q) = Stab(p)$. Type-definability of $Stab(p)$ is clear, so is wideness. The fact that $Stab(p) = Stab'(q)$ is medium follows from Claim 2 and property (A).

Connectedness is proved as in Claim 6 of Theorem 2.10. Finally, the fact that any wide type in $Stab(p)$ lies in $St(p)$ is proved as Claim 7 in Theorem 2.10 replacing $Stab(q)$ there by $Stab'(q)$. \square

The following lemma will be useful later to check that the hypothesis of the theorem are satisfied.

Lemma 2.14. *Assume that μ is left invariant and condition (A) holds. Let q, r be medium and wide types. Let $p \in St(q, r)$ be a wide type and take $(a, b) \in p \times_{nf} p$. Then $\text{tp}(a^{-1}b/M)$ is medium.*

Proof. We show that $a^{-1}b \in St(q)$, i.e., that $aq \cap bq$ is wide. As q is medium, by stability, it is enough to show this for some pair $(a, b) \in p \times_{nf} p$. Take $(a_i : i < \omega)$ an indiscernible sequence in p with $\text{tp}(a_n/Ma_{<n})$ wide and it is enough to show that $a_0q \cap a_1q$ is wide. By assumption $a_0q \cap r$ is wide. As r is medium, by the S1 property, $a_0q \cap a_1q \cap r$ is wide, hence $a_0q \cap a_1q$ is wide as required. \square

3 Groups with f-generics in NTP_2

In this section we will use Theorem 2.10 to prove Theorem 3.18, which is a stabilizer theorem for strong f-generic types in a group G definable in an NTP_2 theory (see Definition 3.3).

We work here with a complete theory T and let \mathcal{U} denote a monster model of T .

We recall the definition of NTP_2 .

Definition 3.1. We say that $\phi(\bar{x}, \bar{y})$ has TP_2 if there are $(a_{lj})_{l,j < \omega}$ in \mathcal{U} and $k \in \omega$ such that:

- (1) $\{\phi(\bar{x}, a_{l,j})_{j \in \omega}\}$ is k -inconsistent for all $l < \omega$.
- (2) For all $f : \omega \rightarrow \omega$, $\{\phi(\bar{x}, a_{l,f(l)}) : l \in \omega\}$ is consistent.

A formula $\phi(\bar{x}, \bar{y})$ is NTP_2 if it does not have TP_2 . The theory T is NTP_2 if no formula has TP_2 .

We will assume throughout this section that T is NTP_2 . Let G be a \emptyset -definable group. Recall that an extension base is a set A such that no $p \in S(A)$ forks over A . We will use the following results (the first three are from [CK12] and the fourth one from [BYC14]).

Fact 3.2. *Let T be an NTP_2 theory and A an extension base.*

- (1) *For any b , there is an A -indiscernible sequence $(b_i : i < \omega)$ such that for any formula $\phi(x; b)$ which divides over A , the partial type $\{\phi(x; b_i) : i < \omega\}$ is inconsistent.*
- (2) *A formula forks over A if and only if it divides over A .*
- (3) *Condition **(F)** is satisfied: given any type p over A , there are $a, b \models p$ such that $\text{tp}(a/Ab)$ and $\text{tp}(b/Aa)$ are non-forking over A .*
- (4) *The ideal of formulas which do not fork over A has the $S1$ property.*

Definition 3.3. A global type $p \in S_G(\mathcal{U})$ is strongly (left) f -generic over A if for all $g \in G(\mathcal{U})$, $g \cdot p$ does not fork over A .

It is strongly bi- f -generic if for all $g, h \in G(\mathcal{U})$, $g \cdot p \cdot h$ does not fork over A .

It is proved in [HP11] that a definable group in an NIP theory is definably amenable (that is, admits a definable G -invariant measure on definable sets) if and only if it admits a strong f -generic type over some model. The theory of definably amenable NIP groups was studied in [HPP08], [HP11] and [CS16] (amongst other papers). In particular, the paper [CS16] characterizes in various ways formulas which extend to strong f -generic types. We generalize here those results to the NTP_2 context, assuming that G admits a strong f -generic type. The proofs are very similar to those in [CS16].

First, we generalize Proposition 5.11 (i) of [HP11], with essentially the same proof.

Lemma 3.4. *If for some model M , G admits a strongly f -generic type over M , then the same is true over any extension base A .*

Proof. We expand the structure by adding a new sort S which, as a set, is a copy of the group G and we put all G -invariant relations on it. So S becomes a homogeneous space for G and any point of S gives rise to a definable bijection between S and G . This expanded structure is NTP_2 , and is conservative: it does not add any definable sets to the main sort. For $A \subseteq \mathcal{U}$, there is a strongly f -generic type over A if and only if the formula $x_S = x_S$ in the expanded structure does not fork over A . (See [HP11, Proposition 5.11] or [Sim15, Lemma 8.19].)

Now assume that $x_S = x_S$ does not fork over some $M \subseteq \mathcal{U}$ and let $A \subseteq \mathcal{U}$ be an extension base. Let \tilde{N} be an $|M|^+$ -saturated model of the expanded theory containing A .

Claim. In this expansion, the type $\text{tp}(M/A)$ does not fork over A .

Proof. Assume it did. Then by definition it implies a disjunction of formulas, each dividing over A . As the expansion is conservative, we may assume that those formulas have parameters in the main sort. But then we can forget about the additional sort and use the fact that $\text{tp}(M/A)$ does not fork over A in the original structure as A is an extension base. \square

There is therefore $M' \equiv_A M$ such that $\text{tp}(M'/\tilde{N})$ does not fork over A . By assumption, there is some $d \in S$ such that $\text{tp}(d/M'\tilde{N})$ does not fork over M' . Then by transitivity of non-forking, $\text{tp}(d/\tilde{N})$ does not fork over A as required. \square

Lemma 3.5. *Let $A \subseteq N$, N is $|A|^+$ -saturated. Assume that $p \in S(\mathcal{U})$ is strongly f -generic over A . Let $a \models p|_N$ and $b \models p|_{Na}$. Then $\text{tp}(ba^{-1}/N)$ extends to a global type, strongly bi- f -generic over A .*

Proof. Let $g, h \in G(N)$. Then $\text{tp}(gb/Na)$ does not fork over A and neither does $\text{tp}(ha/N)$. By transitivity of non-forking, $\text{tp}(gb, ha/N)$ does not fork over A . Hence $\text{tp}(gba^{-1}h^{-1}/N)$ does not fork over A . Since g, h were arbitrary in $G(N)$, this shows that $\text{tp}(ba^{-1}/N)$ is strongly bi- f -generic over A .

Since N is $|A|^+$ -saturated, $\text{tp}(ba^{-1}/N)$ extends to a global type strongly bi- f -generic over A . (This is a closed condition and any finite part of it can be dragged down into N .) \square

We will say that the group G has strong f -generics if it has a strongly f -generic type over some/any extension base. By Lemma 3.5 it would then also have a strong bi- f -generic type over any extension base.

Definition 3.6. Let $\phi(x) \in L(A)$ be a formula. We say that $\phi(x)$ is *f-generic* over A if no (left) translate of $\phi(x)$ forks over A . We say that $\phi(x)$ *G-divides* over A if for some A -indiscernible sequence $(g_i : i < \omega)$ of elements of G , the partial type $\{g_i \cdot \phi(x) : i < \omega\}$ is inconsistent.

Lemma 3.7. *Let A be an extension base and $\phi(x) \in L(A)$. Then $\phi(x)$ is f-generic over A if and only if it does not G-divide over A .*

Proof. If for some $g \in G$, $\phi(g^{-1}x)$ forks over A , then it divides over A and there is an A -indiscernible sequence $(g_i : i < \omega)$ such that $\{\phi(g_i^{-1}x) : i < \omega\}$ is inconsistent. This shows that $\phi(x)$ G-divides over A . Conversely, if $\phi(x)$ G-divides over A as witnessed by $(g_i : i < \omega)$, then $\phi(g_0^{-1}x)$ divides over A . \square

Let $A \subseteq B$ be two extension bases over which $\phi(x; a)$ is defined, then $\phi(x; a)$ G-divides over A if and only if it G-divides over B so the same is true for f-generic. From now on, we drop the “over A ” when talking about f-generic formulas.

Lemma 3.8. *Assume that the formula $\phi(x; b)$ forks over A and that $\text{tp}(g/Ab)$ does not fork over A . Then $\phi(gx; b)$ forks over A .*

Proof. Assume that $\phi(gx; b)$ does not fork over A and let $c \models \phi(gx; b)$ with $c \perp_A Abg$. Then $c \perp_{Ag} Abg$. We also have $g \perp_A Ab$ by hypothesis. By transitivity, $gc \perp_A Ab$. Since $gc \models \phi(x; b)$, we get that $\phi(x; b)$ does not fork over A . \square

Proposition 3.9. *Let A be an extension base, $A \subseteq B$ and $\phi(x) \in L(B)$. Let q be a global type strongly f-generic over A and $g \models q|_B$. Then $\phi(x)$ extends to a global type strongly f-generic over A if and only if $g^{-1} \cdot \phi(x)$ does not fork over A .*

Proof. Assume that $\phi(x)$ does not extend to a global type strongly f-generic over A . Then there are elements g_i , $i < n$ in $G(\mathcal{U})$ and formulas $\phi_i(x; b) \in L(\mathcal{U})$ each forking over A such that $\phi(x) \vdash \bigvee_{i < n} \phi_i(g_i x; b)$. We can assume that g realizes q over $Bb\{g_i\}_{i < n}$. We have then that $\phi(gx) \vdash \bigvee_{i < n} \phi_i(g_i gx; b)$. Now, $\text{tp}(g_i g/Ab)$ does not fork over A for each $i < n$. By Lemma 3.8, this implies that $\phi_i(g_i gx; b)$ forks over A . Hence $\phi(gx) = g^{-1} \cdot \phi(x)$ forks over A .

Conversely, if $\phi(x)$ extends to some global type strongly f-generic over A , then no translate of $\phi(x)$ forks over A and in particular $g^{-1}\phi(x)$ does not fork over A . \square

The previous results combine into the following equivalences.

Proposition 3.10. *Let A be an extension base and assume that there is a global type q strongly f -generic over A . Let $\phi(x) \in L(A)$ and let g realize q over A . The following are equivalent:*

1. $\phi(x)$ is f -generic;
2. $\phi(x)$ does not G -divide over A ;
4. $g^{-1} \cdot \phi(x)$ does not fork over A ;
5. $\phi(x)$ extends to a global type strongly f -generic over A .

As usual, we extend definitions from definable sets to types: we define a type to be f -generic if it contains only f -generic formulas.

Proposition 3.11. *Let A be an extension base and assume that there is a global f -generic type q . Let $\phi(x) \in L(A)$ and let g realize q over A . Then $\phi(x)$ is f -generic if and only if $g^{-1} \cdot \phi(x)$ does not fork over A .*

Proof. If $\phi(x)$ is f -generic, then $g^{-1} \cdot \phi(x)$ does not fork over A by definition.

Conversely, assume that $\phi(x)$ does G -divide and let $(g_i : i < \omega)$ be an A -indiscernible sequence witnessing it. Let $\hat{q} = q|_A$.

Claim. The partial type $\bigcup g_i^{-1} \cdot \hat{q}$ is consistent.

Proof. If not, then there is a formula $\psi(x) \in \hat{q}(x)$ such that $\{g_i^{-1} \cdot \psi(x) : i < \omega\}$ is inconsistent. Then $g_0^{-1} \cdot \psi(x)$ divides over A , contradicting the assumption on q . \square

Let h realize $\bigcup g_i^{-1} \cdot \hat{q}$, so $g_i \cdot h \models \hat{q}$ for each i . Notice that $\{h^{-1}g_i^{-1} \cdot \phi(x) : i < \omega\}$ is still k -inconsistent for some k , and $g^{-1} \cdot \phi(x)$ divides over A as required. \square

Corollary 3.12. *Assume that there is a global f -generic type, then the family μ of non- f -generic formulas is an ideal.*

Proof. Let q be a global f -generic type. Let $\phi(x)$ and $\psi(x)$ be non- f -generic and take M a model over which both are defined. Let $g \models q|_M$ as in the previous proposition. Then $g^{-1} \cdot \phi(x)$ and $g^{-1} \cdot \psi(x)$ both fork over M , hence so does $g^{-1} \cdot (\phi(x) \vee \psi(x))$ —as forking equals dividing over M —which implies that $\phi(x) \vee \psi(x)$ is not f -generic. \square

Question 3.13. *Assume that there is a global f -generic type, then is there a strongly f -generic type?*

Notice that the ideal μ of non-f-generic formulas is \emptyset -invariant and invariant by translations on the left and on the right. It is however not S1 in general. For this we have to work with μ_A .

Assume that G has a strong f-generic type over A . Let μ_A be the ideal of formulas $\phi(x) \in L(\mathcal{U})$ which do not extend to a global type strongly f-generic over A . Then μ_A is A -invariant, left- G -invariant over A . By Proposition 3.10, μ and μ_A agree on $L(A)$.

Lemma 3.14. *The ideal μ_A is S1.*

Proof. Assume that $(a_i : i < \omega)$ is an A -indiscernible sequence such that $\phi(x; a_i)$ extends to a type strongly f-generic over A . Let q be strongly f-generic over A and let g realize q over $Aa_{<\omega}$ such that $(a_i)_{i < \omega}$ is indiscernible over Ag . Then $g^{-1} \cdot \phi(x; a_i)$ is non-forking over A for all i . As the non-forking ideal is S1 in NTP_2 theories, also $g^{-1} \cdot (\phi(x; a_0) \wedge \phi(x; a_1))$ is non-forking over A . By Proposition 3.10, $\phi(x; a_0) \wedge \phi(x; a_1)$ is μ_A -wide. \square

3.1 Stabilizers of strong f-generic types

We will need the following definitions.

Definition 3.15. Let G be a definable group, and M be a model over which G is definable.

We will say that a subset $X \subset G$ is *generic* if finitely many translates cover G .

If H is a type definable (with parameters in M_0) subgroup of G (or more generally an automorphism invariant subgroup), we will say that H has *bounded index in G* if we have that the cardinality of $G(M^*)/H(M^*)$ is smaller than the cardinality of M^* for some saturated model M^* extending M .

Finally, we define G_M^{00} to be the smallest type definable over M subgroup of bounded index and we define G_M^∞ to be the smallest M -invariant subgroup of G of bounded index.

Lemma 3.16. *Let X be an f-generic definable set. Then XX^{-1} is generic.*

Proof. Let $(a_i : i < n)$ be a maximal sequence such that the sets $(a_i X : i < n)$ are disjoint, which must exist by f-genericity of X . Take any $b \in G$. Then for some $i < n$, $bX \cap a_i X \neq \emptyset$. Hence $b \in a_i X X^{-1}$ and $\bigcup_{i < n} a_i X X^{-1} = G$. \square

Lemma 3.17. *Let $H < G$ be a type-definable group. Assume that H is μ -wide (i.e., every definable set containing it is μ -wide), then H has bounded index.*

Proof. Let X be a definable set containing H . Then there is a definable set Y containing H such that $YY^{-1} \subseteq X$. By hypothesis, Y is f -generic and the previous lemma implies that YY^{-1} is generic and therefore X is generic. \square

In the following statement, μ_M is the ideal of formulas which do not extend to a global type, strongly f -generic over M .

Theorem 3.18. *Assume that G has strong f -generics. Let $p \in S_G(M)$ be f -generic.*

Then $G_M^{00} = G_M^\infty = St_{\mu_M}(p)^2 = (pp^{-1})^2$ and $G_M^{00} \setminus St_{\mu_M}(p)$ is contained in a union of non-wide M -definable sets.

Proof. The ideal μ_M is G -invariant (by left multiplication), M -invariant and S1 on G by Lemma 3.14. We can apply Theorem 2.10 with hypothesis (B1) to deduce that $S = (pp^{-1})^2$ is a wide subgroup. As p knows in which G_M^∞ coset it lies, we must have $S \leq G_M^\infty$. On the other hand, by Lemma 3.17, S has bounded index, hence $G_M^{00} \leq S$. It follows that those three subgroups are equal. The last statement also follows from Theorem 2.10. \square

3.2 Definably amenable groups

A definable group G is *definably amenable* if for some (equiv. any) model M , there is a left-invariant Keisler measure on M -definable subsets of G . (See e.g. [Sim15, Chapter 8].)

Fact 3.19 ([Sim15], Lemma 7.5). *Let μ be a measure over M and $(b_i : i < \omega)$ an indiscernible sequence in M . Let $\phi(x; y)$ be a formula and $r > 0$ such that $\mu(\phi(x; b_i)) \geq r$ for all $i < \omega$. Then the partial type $\{\phi(x; b_i) : i < \omega\}$ is consistent.*

Proposition 3.20. *Let G be a definably amenable NTP_2 group, then G has strong f -generics.*

Proof. Fix a model M and μ a G -invariant measure on M -definable sets. Let $M \prec^+ N$ and assume that μ does not extend to a measure over N which is both G -invariant and non-forking over M . By compactness, there are $\epsilon > 0$

and finitely many formulas $\phi_i(x; d)$, $i < n$, each forking over M such that any G -invariant extension $\tilde{\mu}$ of μ satisfies $\bigvee_{i < n} \tilde{\mu}(\phi_i(x; d)) > \epsilon$. Take $(d_j : j < \omega)$ an indiscernible sequence in $\text{tp}(d/M)$ which witnesses dividing as given by Fact 3.2, (1). The condition that $\tilde{\mu}$ extends μ and is G -invariant is invariant under $\text{Aut}(N/M)$, therefore for every j , we also have $\bigvee_{i < n} \tilde{\mu}(\phi_i(x; d_j)) > \epsilon$. So up to taking a subsequence, for some $i < n$, we have $\bigwedge_{j < \omega} \tilde{\mu}(\phi_i(x; d_j)) > \epsilon$. But this contradicts Fact 3.19 and the property of $(d_j)_{j < \omega}$. \square

Corollary 3.21. *Any solvable or pseudofinite NTP_2 group has strong f -generics.*

4 PRC fields

A field M of characteristic zero is *pseudo real closed (PRC)* if M is existentially closed (relative to the language of rings) in every totally real regular extension N of M . Equivalently, if given any absolutely irreducible variety V defined over M , if V has a simple \overline{M}^r -rational point for every real closure \overline{M}^r of M , then V has an M -rational point.

Prestel showed in Theorem 4.1 of [Pre82] that the class of PRC fields is axiomatizable in the language of fields. We have the following properties of PRC fields.

Fact 4.1. *Let M be a PRC field.*

- (1) [Pre82, Proposition 1.4] *If $<$ is an order on M , then M is dense in $(\overline{M}^r, \overline{<}^r)$, the real closure of M respect to the order $<$.*
- (2) [Pre82, Proposition 1.6] *If $<_i$ and $<_j$ are different orders on M , then $<_i$ and $<_j$ induce different topologies.*

In this section we are interested in the class of bounded PRC fields. A field M is *bounded* if for any integer n , M has finitely many extensions of degree n . This implies in particular that all the orders which make M into an ordered field are definable ([Mon16, Lemma 3.5]), and that there are finitely many of those.

4.1 Preliminaries on bounded PRC fields

We fix a bounded PRC field K which is not algebraically closed and a countable elementary substructure K_0 of K . So there is $n \in \mathbb{N}$ such that K has

exactly n distinct orders which are moreover definable (see Remark 3.2 of [Mon16]). Let $\{<_1, \dots, <_n\}$ be the orders on K . If $n = 0$, then K is a PAC field, so we suppose from now on that $n \geq 1$.

We will work over K_0 , thus we denote by $\mathcal{L}_{\text{ring}}$ the language of rings with constant symbols for the elements of K_0 , $\mathcal{L}_{\text{ring}}^{(i)} := \mathcal{L}_{\text{ring}} \cup \{<_i\}$ and $\mathcal{L} := \mathcal{L}_{\text{ring}} \cup \{<_1, \dots, <_n\}$. We let $T_{\text{prc}} := Th_{\mathcal{L}_{\text{ring}}}(K) = Th_{\mathcal{L}}(K)$. By Corollary 3.6 of [Mon16], T_{prc} is model complete. If M is a model of T_{prc} , we denote by $M^{(i)}$ the real closure of M with respect to $<_i$.

The following is a direct consequence of the “Approximation Theorem for V -topologies” ([PZ78, Theorem 4.1]), and of the Fact 4.1.

Fact 4.2. *Let $(M, <_1, \dots, <_n)$ be a model of T_{prc} . Let A be a subset of M and for every order $<_i$ let $p^{(i)}$ be a quantifier-free $\mathcal{L}_{\text{ring}}^{(i)}$ -type in $M^{(i)}$ (so a consistent set of polynomial $<_i$ -inequalities). Then $\bigcup_{i=1}^n p^{(i)}$ is a consistent type in \mathcal{L} .*

Notice that the quantifier free \mathcal{L} -types all have the same form as the conclusion of Fact 4.2. We have the following amalgamation theorems for types:

Fact 4.3 ([Mon16], Theorem 3.21). *Let $(M, <_1, \dots, <_n)$ be a model of T_{prc} . Let $E = \text{acl}(E) \subseteq M$. Let a_1, a_2, c_1, c_2 be tuples of M such that $E(a_1)^{\text{alg}} \cap E(a_2)^{\text{alg}} = E^{\text{alg}}$ and $\text{tp}_{\mathcal{L}}(c_1/E) = \text{tp}_{\mathcal{L}}(c_2/E)$. Assume that there is c ACF-independent of $\{a_1, a_2\}$ over E realizing $\text{qftp}_{\mathcal{L}}(c_1/E(a_1)) \cup \text{qftp}_{\mathcal{L}}(c_2/E(a_2))$. Then $\text{tp}_{\mathcal{L}}(c_1/Ea_1) \cup \text{tp}_{\mathcal{L}}(c_2/Ea_2) \cup \text{qftp}_{\mathcal{L}}(c/E(a_1, a_2))$ is consistent.*

We now recall some other model theoretic properties of T_{prc} .

Fact 4.4 ([Mon16], Theorem 4.21). *The theory T_{prc} is NTP_2 .*

Fact 4.5 ([Mon16], Theorem 4.35). *In T_{prc} , all sets are extensions bases and forking equals dividing.*

4.2 The multi-topology

Definition 4.6. Let $(M, <_1, \dots, <_n)$ be a model of T_{prc} , $A \subseteq M$ and let $X \subseteq M^m$ be $\mathcal{L}_{\text{ring}}(A)$ -definable. Then $\dim(X) = \max\{\text{trdeg}(\bar{x}/A) : \bar{x} \in X\}$. This is a good notion of dimension, since $\text{acl}(A) = \text{dcl}(A) = A^{\text{alg}} \cap M$ ([Mon16, Lemma 2.6]). We will say that $\bar{a} \in X$ is a *generic point of X over A* if $\dim(X) = \text{trdeg}(\bar{a}/A)$.

Definition 4.7. (Multi-topology) Let $(M, <_1, \dots, <_n)$ be a model of T_{prc} . Denote by τ_i the topology induced in M by the order $<_i$. By Fact 4.1 (2), if $i \neq j$, then $\tau_i \neq \tau_j$.

A definable subset of M of the form $I = \bigcap_{i=1}^n (I^i \cap M)$ with I^i a non-empty $<_i$ -open interval in $M^{(i)}$ is called a *multi-interval*.

Notice that by Fact 4.2 every multi-interval is non-empty and if I is a multi-interval, then each I^i is $<_i$ -dense in I .

We define the *multi-topology* τ as the topology in M generated by the multi-intervals and τ^m its product topology in M^m . Observe that if V is τ_i -open, then it is τ -open. We call a *multi-box in M^m* a set of the form $C = \bigcap_{i=1}^n (C^i \cap M^m)$, with C^i an $<_i$ -box in $M^{(i)}$.

We extend the definition of (j_1, \dots, j_r) -cells for real closed fields (see Definition 2.3 of [vdD98]) to find a definition of multi-cells in the bounded PRC-field context.

Definition 4.8. (Multi-cells) Let $r \in \mathbb{N}$ and let (j_1, \dots, j_r) be a sequence of zeros and ones of length r .

A (j_1, \dots, j_r) -multi-cell is definable subset C of M^r such that for every i there is a (j_1, \dots, j_r) -cell C^i in $M^{(i)}$ and

$$C = \bigcap_{i=1}^n (C^i \cap M^r).$$

A *multi-cell in M^r* is a (j_1, \dots, j_r) -multi-cell, for some (j_1, \dots, j_r) .

Observe that the (1)-multi-cells are multi-intervals and any multi-box is a $(1, \dots, 1)$ -multi-cell.

Notice also that the *open multi-cells in M^r* (or cells which are open subsets of M^r) are precisely the $\left(\underbrace{1, \dots, 1}_r\right)$ -multi-cells.

Lemma 4.9. Let $m \in \mathbb{N}$ and let (i_1, \dots, i_m) and (j_1, \dots, j_m) be two different sequences of zeros and ones of length m . Let $C^i \in M^{(i)}$ be a (i_1, \dots, i_m) -cell and let $C^j \in M^{(j)}$ be a (j_1, \dots, j_m) -cell. Then $\dim(C^i \cap C^j \cap M^m) < \min\{\dim(C^i), \dim(C^j)\}$.

Proof. Let $r_i = \dim(C^i)$ and $r_j = \dim(C^j)$. Suppose that there is $\bar{a} = (a_1, \dots, a_m) \in C$ such that \bar{a} is a generic point of C^i and C^j . Let $X_i = \{a_k : i_k = 0\}$, $X_j = \{a_k : j_k = 0\}$. Then $r_i = m - |X_i|$ and $r_j = m - |X_j|$. Observe that if $a_k \in X_i \cup X_j$, then $a_k \in \text{acl}(a_1, \dots, a_{k-1})$. It follows that $\dim(C) \leq m - |X_i \cup X_j|$. Since $(i_1, \dots, i_m) \neq (j_1, \dots, j_m)$, $|X_i \cup X_j| > \max\{|X_i|, |X_j|\}$. Thus $\dim(C) \leq m - |X_i \cup X_j| < \min\{m - |X_i|, m - |X_j|\} = \min\{r_i, r_j\}$. \square

It follows that for an intersection of two r -dimensional cells to have dimension r , one needs that both cells have the same sequences of 0's and 1's.

Theorem 4.10. *Let $(M, <_1, \dots, <_n)$ be a model of T_{prec} ; let $A \subseteq M$, and $r \in \mathbb{N}$. Let $D \subseteq M^r$ be an $\mathcal{L}(A)$ -definable set in M . Then there are $m \in \mathbb{N}$, and C_1, \dots, C_m with $C_j = \bigcap_{i=1}^n (C_j^i \cap M^r)$ a multi-cell in M^r such that:*

- (1) $D \subseteq \bigcup_{j=1}^m C_j$;
- (2) $D \cap C_j$ is τ^r -dense in C_j , for all $1 \leq j \leq m$;
- (3) for all $1 \leq i \leq n$ and $1 \leq j \leq m$, C_j^i is quantifier-free $\mathcal{L}_{\text{ring}}^{(i)}(A)$ -definable in $M^{(i)}$;
- (4) for all $1 \leq i \leq n$ and $1 \leq j \leq m$, the set $C_j^i \cap M^r$ is $\mathcal{L}_{\text{ring}}^{(i)}(A)$ -definable in M .

Proof. The proof is by induction on the dimension of D . The case $\dim(D) = 1$ follows from [Mon16, Theorem 3.13]. Suppose that $\dim(D) = d$. As in Theorem 3.13 [Mon16] using model completeness of T_{prec} we can suppose that there is an absolutely irreducible variety W defined over $\text{acl}(A)$ such that:

$$M \models \forall x_1, \dots, x_r ((x_1, \dots, x_r) \in D) \longleftrightarrow (\exists \bar{y} (x_1, \dots, x_r, \bar{y}) \in W^{\text{sim}}(M)),$$

where $W^{\text{sim}}(M) = \{\bar{x} \in W(M) : \bar{x} \text{ is a simple point of } W\}$.

Let $d = |\bar{y}|$, for each $i \in \{1, \dots, n\}$ we define:

$$A_i := \{(x_1, \dots, x_r) \in (M^{(i)})^r : \exists \bar{y} \in (M^{(i)})^d \text{ s.t. } (x_1, \dots, x_r, \bar{y}) \in W^{\text{sim}}(M^{(i)})\}.$$

So A_i is $\mathcal{L}_{\text{ring}}^{(i)}(A)$ -definable and $D \subseteq A_i$. By cell decomposition in $M^{(i)}$, there are $k_i \in \mathbb{N}$, $<_i$ -cells $C_1^i, \dots, C_{k_i}^i$ and X such that:

- (1) the sets $C_1^i, \dots, C_{k_i}^i, X^i$ are quantifier free $\mathcal{L}_{\text{ring}}^{(i)}(A)$ -definable in $M^{(i)}$;
- (2) $\dim(C_j) = d$, for all $j \in \{i_1, \dots, i_{k_i}\}$;
- (3) $\dim(X^i) < d$,

$$(4) \ A_i = \bigcup_{j=1}^{k_i} C_j \cup X^i.$$

Let $X = \bigcup_{i=1}^n (X^i \cap M^r)$ and let

$$J := \{\sigma : \{1, \dots, n\} \rightarrow \mathbb{N} \mid 1 \leq \sigma(i) \leq k_i\}.$$

For all $\sigma \in J$, let $C_\sigma := \bigcap_{i=1}^n (C_{\sigma(i)}^i \cap M^r)$, so $D \subseteq \bigcup_{\sigma \in J} C_\sigma \cup X$. We are interested in C_σ of maximal dimension d , so let

$$J' := \{\sigma \in J : \dim(C_\sigma) = d\}.$$

Let $\sigma \in J'$. By Lemma 4.9 all the cells $C_{\sigma(i)}^i$ must have the same sequences of 0's and 1's and therefore C_σ is a multi-cell in M^r .

Claim. For all $\sigma \in J'$, $D \cap C_\sigma$ is τ^r -dense in C_σ .

Proof. Fix $\sigma \in J'$. Let U_σ be a multi-box in M^r such that $V := U_\sigma \cap C_\sigma \neq \emptyset$, we need to show that $V \cap D \neq \emptyset$. Let $z \in V$. Then $z \in A_i$ for all $i \in \{1, \dots, n\}$. So there is $y^{(i)} \in (M^{(i)})^d$, such that $(z, y^{(i)})$ is a simple point of W . By Fact 4.2 we can find $(z_0, \bar{y}_0) \in W(M)$ such that (z_0, \bar{y}_0) is arbitrary $<_i$ -close to $(z, y^{(i)})$ for all $i \in \{1, \dots, n\}$, in particular we can find $z_0 \in V \cap D$. \square

Let $Y = X \cup \bigcup_{\sigma \in J \setminus J'} C_\sigma$, so Y is an $\mathcal{L}(A)$ -definable set and $\dim(Y) < d$.

Then $D \subseteq \bigcup_{\sigma \in J'} C_\sigma \cup Y$ and each C_σ satisfy (2), (3) and (4) of the theorem.

Since $\dim(Y) < d$, by induction hypothesis we can apply the statement of the theorem to Y instead of D , which completes the proof. \square

Definition 4.11. Let $(M, <_1, \dots, <_n)$ be a model of T_{prc} and $D \subseteq M^r$ a definable set. Denote by \overline{D} the closure of D for the τ^r -topology. Observe that $\overline{D} = \bigcap_{i=1}^n \overline{D}^{\tau_i}$, where \overline{D}^{τ_i} is the closure of D for the τ_i -topology.

If $X \subseteq M^r$ is a definable set and C_1, \dots, C_m are the multi-cells obtained by Theorem 4.10, then $\bigcup_{j=1}^m C_j \subseteq \overline{D}$. This implies the following corollary.

Corollary 4.12. *Let $(M, <_1, \dots, <_n)$ be a model of T_{prc} , let $A \subseteq M$, and $r \in \mathbb{N}$. Let $D \subseteq M^r$ be an $\mathcal{L}(A)$ -definable set in M . Then there are $m \in \mathbb{N}$, and C_1, \dots, C_m with $C_j = \bigcap_{i=1}^n (C_j^i \cap M^r)$ a multi-cell in M^r such that: $\overline{D} = \bigcup_{j=1}^m C_j$ and such that for all $1 \leq i \leq n$, C_j^i is quantifier-free $\mathcal{L}_{ring}^{(i)}(A)$ -definable in $M^{(i)}$, for all $1 \leq j \leq m$.*

Proof. The set \overline{D} is $\mathcal{L}(A)$ -definable (so was D) and by Theorem 4.10 there are C_1, \dots, C_m multi-cells in M^r such that $\overline{D} \subseteq \bigcup_{j=1}^m C_j \subseteq \overline{(\overline{D})} = \overline{D}$. So

$$\overline{D} = \bigcup_{j=1}^m C_j. \quad \square$$

Definition 4.13. A theory T in a language containing the language of rings and which contains the theory of fields, is *algebraically bounded* if, given any formula $\phi(\bar{x}, y)$, there are polynomials $f_1(\bar{x}, y), \dots, f_n(\bar{x}, y) \in \mathbb{Z}[\bar{x}, y]$ such that, whenever K is a model of T and \bar{a} is a tuple of elements of K such that $\phi(\bar{a}, K) := \{y \in K : \phi(\bar{a}, y)\}$ is finite, then there is an index i such that the polynomial $f_i(\bar{a}, y)$ is not identically 0 on K and $\phi(\bar{a}, K)$ is contained in the set of roots of $f_i(\bar{a}, y) = 0$.

Corollary 4.14. *The theory T_{prc} is algebraically bounded.*

Proof. Directly from Theorem 4.10. \square

Notation. Let M be a structure and let $D \subseteq M^r$ be a definable set. Let $k < r$. We define $\pi_k^M(D) := \{(x_1, \dots, x_k) \in M^k : M \models \exists x_{k+1}, \dots, x_r (x_1, \dots, x_r) \in D\}$.

For $\bar{\alpha} \in \pi_k^M(D)$, define $D_{\bar{\alpha}}^M := \{\bar{y} \in M^{r-k} : (\bar{\alpha}, \bar{y}) \in D\}$. Define $D^M(a_1, \dots, a_k) := \{(a_1, \dots, a_k, x_{k+1}, \dots, x_r) : M \models (a_1, \dots, a_k, x_{k+1}, \dots, x_r) \in D\}$. We omit M when the structure is clear.

4.3 Expansion by externally definable multi-cells

Here we will show that expanding a bounded PRC field with certain *externally definable* sets has elimination of quantifiers, analogous to results in [BP98] and [She09].

Definition 4.15. Let T be a theory and let M be a model of a theory T . An *externally definable subset of M^k* is an $X \subseteq M^k$ that is equal to $\varphi(N^k, d) \cap M^k$ for some formula φ and d in some $N \succeq M$.

Definition 4.16. Let $(M, <_1, \dots, <_n)$ be a model of T_{prc} . We say that $C = \bigcap_{i=1}^n C^i \cap M^r$ is an *externally definable multi-cell in M^r* if for $i \in \{1, \dots, n\}$, C^i is the trace on $(M^{(i)})^r$ of a cell defined with exterior parameters. We say that C is a *multi-cell externally $\mathcal{L}(N)$ -definable* if $N \succeq M$ and for each $i \in \{1, \dots, n\}$, there is $\phi_i(\bar{x}) \in \mathcal{L}(N^{(i)})$ such that $C^i = \phi_i(M)$.

Baisalov and Poizat prove in [BP98] that the theory resulting in expanding the language of any o-minimal structure with externally definable sets has elimination of quantifiers. This was generalized by Shelah to all NIP theories in [She09].

Proposition 4.17. *Let R be a model of RCF in the ring language $\mathcal{L}_{\text{ring}}$ and consider the expansion R^{Sh} of R obtained by naming all externally definable sets. Then any definable subset of R^{Sh} can be written as a finite union of sets of the form $U \cap D$, where U is an open externally definable subset and D is $\mathcal{L}_{\text{ring}}$ -definable.*

Proof. By [BP98] the structure R^{Sh} is weakly o-minimal, so it makes sense to consider dimensions of definable sets. Let $X \subseteq R^n$ be definable in R^{Sh} . We prove the result by induction on the dimension of X . If X has dimension 0, then it is finite, and the result follows.

For the inductive case, we can write X as the union of an open set and a set of lower dimension, so we can assume that X has dimension $d < n$. Let $\pi : R^n \rightarrow R^d$ be a coordinate projection such that $\pi(X)$ has non-empty interior (see Theorem 4.11 of [MMS00]). Then again writing $\pi(X)$ as the union of an open set and a set of smaller dimension, we may assume that $\pi(X)$ is open. For each $\bar{a} \in \pi(X)$, the fiber $X_{\bar{a}}$ is finite. By decomposing X further, we may assume that it has always exactly one element. So X is the graph of a function from $U := \pi(X)$ to R^{n-d} .

Let $R^{Sh} \prec R'$ be a sufficiently saturated elementary extension. Then by honest definitions ([Sim15, Chapter 3]), there is an $\mathcal{L}_{\text{ring}}(R')$ -definable set

$X' \subseteq R'$ such that $X'(R') \subseteq X(R')$ and $X'(R) = X(R)$. Hence X' is also the graph of a function from some R' -definable set V to R^{n-d} , with $V(R) = U(R)$. As we are working in RCF, up to decomposing V in finitely many R' -definable sets, we may assume that f' is the function sending a point $\bar{a} \in V$ to the k -th solution of $P(\bar{b}, \bar{a}, \bar{Y})$, where $P(\bar{b}, \bar{T}, \bar{Y})$ is a polynomial with coordinates $\bar{b} \in R'$. Since by hypothesis, $P(\bar{b}, \bar{T}, \bar{Y})$ has a solution in R for each \bar{a} in the open set U , P is definable over R . This implies that X coincides on U with the graph Γ of an R -definable function. Then $X = U \times R^{n-d} \cap \Gamma$ has the required form. \square

We now aim to show that the expansion of a bounded PRC field in \mathcal{L} by externally definable multi-cells has elimination of quantifiers and is NTP₂.

Proposition 4.18. *Let $(M, <_1, \dots, <_n)$ be a model of T_{prc} . Let $A \subseteq M$ and let $D \subseteq M^r$ be $\mathcal{L}(A)$ -definable. Then there are $m \in \mathbb{N}$ and C_1, \dots, C_m multi-cells in M^r , $\mathcal{L}(A)$ -definable such that $D \subseteq \bigcup_{j=1}^m C_j$ and such that for every $\bar{x} \in \pi_{r-1}(D \cap C_j)$ the fiber $D_{\bar{x}}$ is τ -dense in $(C_j)_{\bar{x}}$.*

Proof. Notice that if $D = D_1 \cup D_2$ and the theorem is known for D_1 and D_2 , then it follows for D by taking a common refinement of the two cell decompositions obtained for D_1 and D_2 .

Let D be a definable set. By Theorem 4.10 for any $\bar{x} \in \pi_{r-1}(D)$ there are $k_{\bar{x}}, U_1, \dots, U_{k_{\bar{x}}}$ multi-intervals in M and a finite set $B_{\bar{x}}$ such that $D_{\bar{x}} \subseteq \bigcup_{j=1}^{k_{\bar{x}}} U_{\bar{x},j} \cup B_{\bar{x}}$, and such that $D_{\bar{x}}$ is τ -dense in $U_{\bar{x},j}$, for all $j \in \{1, \dots, k_{\bar{x}}\}$. By

definition of multi-intervals $U_{\bar{x},j} = \bigcap_{i=1}^n U_{\bar{x},j}^i \cap M$, where $U_{\bar{x},j}^i$ is a $<_i$ -interval in $M^{(i)}$.

For all $m_1, m_2 \in \mathbb{N}$, let $A_{m_1, m_2} := \{\bar{x} \in \pi_{r-1}(D) : k_{\bar{x}} = m_1 \text{ and } |B_{\bar{x}}| = m_2\}$. By compactness there are only finitely many (m_1, m_2) for which A_{m_1, m_2} is non empty.

Then A_{m_1, m_2} is definable with the same parameters as D , and $\pi_{r-1}(D) = \bigcup_{(m_1, m_2)} A_{m_1, m_2}$ (a finite union). Since $D = \bigcup_{(m_1, m_2)} \pi_{r-1}^{-1}(A_{m_1, m_2})$, it is enough to show that each $\pi_{r-1}^{-1}(A_{m_1, m_2})$ can be decomposed according to the conclusion of the theorem, so assume that $D = \pi_{r-1}^{-1}(A_{m_1, m_2})$ for some (m_1, m_2) .

Let $i \in \{1, \dots, n\}$. For $s \in \{1, 2, 3\}$, let $f_{s,j}^i(x) : A_{m_1, m_2} \mapsto M^{(i)}$ such that:

- (1) $f_{1,j}^i(\bar{x}) = y$ if and only if y is the “ $<_i$ -smallest extremity in $M^{(i)}$ ” of the $<_i$ -interval $U_{\bar{x},j}^i$.
- (2) $f_{2,j}^i(\bar{x}) = y$ if and only if y is the “ $<_i$ -largest extremity in $M^{(i)}$ ” of the $<_i$ -interval $U_{\bar{x},j}^i$.
- (3) $f_{3,j}^i(\bar{x}) = y$ if and only if y is the j -th point in $B_{\bar{x}}$ in the order $<_i$.

As the structure is algebraically bounded (see Corollary 4.14), there is a definable partition of the base $A_{m_1, m_2} = \bigcup_{t < p} X_t$ such that on each X_t , each of the functions $f_{s,j}^i$ coincides with a $<_i$ -semi-algebraic function. Decreasing D further, we may assume that $p = 1$ and that all the functions $f_{s,j}^i$ are semi-algebraic.

Now, let

$$C_j := \{(\bar{x}, y) \in M^r : f_{1,j}^i(\bar{x}) < y < f_{2,j}^i(\bar{x}), \text{ for all } i\}$$

and

$$C_j^0 := \{(\bar{x}, y) \in M^r : f_{3,j}^1(\bar{x}) = y\}.$$

Then $D \subseteq \bigcup_j C_j \cup \bigcup_j C_j^0$ and this decomposition has the required properties. \square

Definition 4.19. Let \mathcal{U} be a monster model of T_{prc} . Let M be a model of T_{prc} . Let $N \succeq M$ such that N is $|M|^+$ -saturated. Then $N^{(i)} \succeq N$, and $N^{(i)}$ is $|M^{(i)}|^+$ -saturated, for all $i \in \{1, \dots, n\}$.

Let $\mathcal{L}^* = \mathcal{L} \cup \{R_C(\bar{x}) : C \text{ is a multi-cell externally } \mathcal{L}(N)\text{-definable}\} \cup \{P_D(\bar{x}) : D \text{ is } \mathcal{L}(M)\text{-definable}\}$. We define M_N to be the structure in the language \mathcal{L}^* whose universe is M and where each R_C and each P_D are interpreted as:

- (1) for every $\bar{a} \in M$, $M_N \models R_C(\bar{a})$ if and only if $\mathcal{U} \models \bar{a} \in C$,
- (2) for every $\bar{a} \in M$, $M_N \models P_D(\bar{a})$ if and only if $M \models \bar{a} \in D$

Theorem 4.20. *The structure M_N admits elimination of quantifiers.*

Proof. Let C be an externally $\mathcal{L}(N)$ -definable multi-cell and D an $\mathcal{L}(M)$ -definable set, both inside some M^r . Let π be the projection to the first $r - 1$ coordinates. It is enough to show that $\pi(C \cap D)$ is quantifier-free definable in M_N .

First, write $C = \bigcap_{i=1}^n C^i \cap M^r$, where each C^i is an externally definable multi-cell in $(M^{(i)})^r$. By Proposition 4.17, we can write each C^i as a finite union of sets of the form $U^i \cap D^i$, where U^i is an externally definable open subset of $(M^{(i)})^r$ and D^i is definable in $M^{(i)}$. Then the trace of U^i on M^r is also open by density of M^r in $(M^{(i)})^r$ and the trace of D^i on M^r is definable in M . The result we want to prove is stable under taking finite unions, so we may assume that $C^i = U^i \cap D^i$ and then by integrating D^i into D , we may assume that $C = C^i \cap M^r$ is open in M^r .

By Proposition 4.18, we may assume that D is τ -dense in some multi-cell C_* which contains it and such that if $\bar{x} \in \pi(D)$, then $D_{\bar{x}}$ is τ -dense in the fiber $(C_*)_{\bar{x}}$. As C is τ -open, for any $\bar{x} \in \pi(D)$, if the fiber $(C \cap C_*)_{\bar{x}}$ is non-empty, then it is open in $C_{\bar{x}}$ and thus also $(D \cap C)_{\bar{x}}$ is non-empty. Therefore $\pi(D \cap C) = \pi(D) \cap \pi(C \cap C_*)$ is quantifier-free definable in M_N . \square

Corollary 4.21. *The structure M_N is NTP_2 .*

Proof. This follows from Theorem 6.4 proved in appendix. \square

5 Type-definable subgroups of algebraic groups

In the following proposition, by a *definable ideal*, we mean an ideal μ such that for any $\phi(x; y)$, the set $\{b : \phi(x; b) \in \mu\}$ is definable.

Proposition 5.1. *Let G be a definable group equipped with a definable (left-) G -invariant S1 ideal μ . Let $H \leq G$ be a type-definable subgroup of G which is μ -wide, then H is the intersection of definable subgroups of G .*

Proof. The proof follows that of Lemma 6.1 in [HP94], but since the contexts are not exactly the same we include it for completeness.

Write $H = \bigcap_{n < \omega} H_n$, where each H_n is definable, stable under inverse and $H_{n+1} \cdot H_{n+1} \subseteq H_n$. Let $\delta_n(x; y) = x \in G \wedge y \in G \wedge xH_n \cap yH_n \notin \mu$. Then δ_n is a definable, stable (as μ is S1), G -invariant relation. Let $S_{\delta_n, H}$ be the set of global δ_n -types (in variable x) which are consistent with H . By stability, all δ_n -types are definable. Recall that an element of $S_{\delta_n, H}$ is generic if every set in it covers G in finitely many translates. By Lemma 5.16 in [HP94], there are finitely many generic types in $S_{\delta_n, H}$.

Let Q_n be the definable set $\{b \in H_0 : \delta_n(x; b) \text{ is in all generic types of } S_{\delta_n, H}\}$.

Claim. $H = \bigcap Q_n$.

Proof. Let $a \in H$, then for $b \in H$ realizing a generic type of $S_{\delta_n, H}$ over a , $H \subseteq aH_n \cap bH_n$, hence $aH_n \cap bH_n \notin \mu$ and $a \in Q_n$.

Conversely, let $a \in \bigcap Q_n$ and take $b \in H$ generic over a as above. By definition of Q_n , we have $aH_n \cap bH_0 \notin \mu$ for all n so that in particular it is non-empty. Hence by compactness, $aH \cap bH$ is non-empty, so $a \in H$. \square

Claim. $HQ_n \subseteq Q_n$.

Proof. Let $a \in H$ and $b \in Q_n$. Let c generic over a, b . We need to show that $abH_n \cap cH_n \notin \mu$. By invariance, this is equivalent to $bH_n \cap a^{-1}cH_n \notin \mu$. But $a^{-1}c$ realizes a generic over b , hence this follows from the fact that $b \in Q_n$. \square

Finally, let $G_n = \{x \in H_n : xQ_n \subseteq Q_n \wedge x^{-1}Q_n \subseteq Q_n\}$. Then G_n is a subgroup and $H \subseteq G_n \subseteq H_n$, so $H = \bigcap G_n$. \square

Lemma 5.2. *Let (G, \star) be an algebraic group in an \aleph_1 -saturated real closed field $(R, <)$. Then there is an externally definable $<$ -open subgroup $H \leq G$ which has an invariant definable type in the Shelah expansion R^{Sh} , where we expand the language to include all the R^* -definable subsets of R for some saturated $R^* \succeq R$.*

Proof. As in [HPP08, Proposition 7.8], we identify a small neighborhood of e in G with a neighborhood of zero in R^n . If we let ϵ be infinitesimal with respect to R , then we have

$$|x \star y - (x + y)| \leq C|(x, y)|^2$$

for some $C \in R$ and all $|x|, |y| \leq \epsilon$. Let U be the convex set of infinitesimals with respect to R . Then $H = \{\bar{x}, x_i \in U \text{ for all } i\}$ is a subgroup of G .

The set U is definable using parameters in R^* , so U is defined by a predicate \tilde{U} in R^{ext} and therefore H is also defined by a predicate \tilde{H} .

Let \tilde{V} denote the set of elements x in R such that $x \geq 1/n$ for some $0 < n < \omega$, which by compactness and saturation is also the trace in R of an R^* -definable set, so it is definable in R^{Sh} .

Let $p(x_1, \dots, x_n)$ be the type in \tilde{H} saying that x_1 is as large as possible in \tilde{U} , and for all $k > 1$, x_k/x_{k-1} is infinitely small in \tilde{V} . Using weak o-minimality of R^{ext} we know that p determines a (definable) complete type.

We will show that p is \tilde{H} -invariant, so that $\tilde{H}(R)$ and p satisfy the statement of the lemma. Let $\bar{a} = (a_1, \dots, a_n) \in \tilde{H}(R^*)$ and let \bar{b} realize p over R^* . We have to show that $\bar{y} := \bar{a} \star \bar{b}$ realizes p over R^* .

All coordinates of \bar{a} and all b_k^2 , are infinitesimal with respect to each b_k , so $\bar{a} \star \bar{b} = \bar{a} + \bar{b} + \bar{\epsilon}$, where $|\bar{\epsilon}| \leq C \cdot b_1^2$. Now $y_1 = b_1 + a_1 + \epsilon_1$, $a_1 \in R^*$, b_1 as large as possible in U and $|\epsilon_1| \leq b_1^2$ which is much less than b_1 , so $\text{tp}(y_1/R^*) \in U$ satisfies $\text{tp}(b_1/R^*)$.

In the same way, we have

$$\frac{y_k}{y_{k-1}} = \frac{b_k + a_k + \epsilon_k}{b_{k-1} + a_{k-1} + \epsilon_{k-1}}$$

hence

$$\frac{1/2b_k}{2b_{k-1}} \leq \frac{y_k}{y_{k-1}} \leq \frac{2b_k}{1/2b_{k-1}}$$

from which it follows that y_k/y_{k-1} realizes over R the type of an infinitesimally small element in \tilde{V} . So \bar{y} realizes p , as required. \square

Proposition 5.3. *Let M be a model of T_{prc} . Let H be an algebraic group definable in M , let $K \leq H$ be a type definable subgroup and $L = \overline{K}$. Then K has bounded index in L , and L/K with the logic topology is profinite.*

Proof. Let \overline{K}^z be the Zariski closure of K . Then \overline{K}^z is an algebraic subgroup of H , \overline{K}^z is type-definable and $\dim(\overline{K}^z) = \dim(K)$.

So replacing H by \overline{K}^z we can suppose that $\dim(H) = \dim(K) := m$. Observe that K has bounded index in L .

Let $N \succ M$ be $|M|^+$ -saturated. We now work in the structure M_N defined in Definition 4.19. It is NTP₂ by Corollary 4.21. Suppose we have n -definable orders. For each i , we will define, in the ordered $<_i$ -ring language $\mathcal{L}^{(i)}$ (using externally definable sets) a definable set V^i , and a type p^i in $N^{(i)}$ as follows.

For each $i \in \{1, \dots, n\}$, let V^i be the \mathcal{L}^{ext} -definable subgroup of H , and let p^i be the invariant \mathcal{L}^{ext} -definable type given by Lemma 5.2. So V^i is the trace of an $N^{(i)}$ -definable set in $M^{(i)}$.

Let $V := \bigcap_{i=1}^n V^i \cap M$, and let $p := \bigcup_{i=1}^n p^i$. By Fact 4.2, $V \neq \emptyset$ and p is finitely consistent in M .

We have that V is an externally definable set in M , and each p^i is definable in $(N^{(i)})^{\text{ext}}$, so p is a definable partial type in M_N . In a similar way we also obtain that p is V -invariant.

So V is a τ -open definable subgroup of L , and since K is τ -dense in L , all the cosets intersect K and we obtain that $V/V \cap K \cong L/K$.

We define an ideal μ over V by $X \in \mu$ if $\overline{X} \notin p$. This ideal is definable and V -invariant.

Claim. μ is $S1$ over V .

Proof. If X is a definable set, by Theorem 4.10 and Corollary 4.12 it follows that $\overline{X} \in p$ if and only if $X \cup p$ is consistent. Let $\phi(x, y)$ be a formula and let $(a_j)_{j \in \omega}$ be indiscernible over V such that $\phi(x, a_j) \notin \mu$, for all $j \in \omega$. Then all of the formulas $\overline{\phi(x, a_j)}$ are in p , and for each j we have that $\phi(x, a_j) \cup p$ is consistent.

Let c_1 and c_2 be such that $c_1 \models \phi(x, a_1) \cup p$, and $c_2 \models \phi(x, a_2) \cup p$, and such that c_1 and c_2 are algebraically independent over $\{a_1, a_2\}$. By Fact 4.3 $\text{tp}(c_1/a_1) \cup \text{tp}(c_2/a_2) \cup p$ is consistent. It follows that $\phi(x, a_1) \cup \phi(x, a_2) \cup p$ is consistent, and by τ -completeness of p we have

$$\overline{\phi(x, a_1) \cap \phi(x, a_2)} \in p.$$

By indiscernibility

$$\overline{\phi(x, a_i) \cap \phi(x, a_j)} \in p$$

for all $i \neq j$, so $\phi(x, a_i) \cap \phi(x, a_j) \notin \mu$, for all $i \neq j$. \square

Now, $\overline{V \cap K} \in p$ so that $V \cap K$ is μ -wide. It follows by Theorem 5.1 that $V \cap K$ is an intersection of definable groups. Hence $V/V \cap K$ with the \mathcal{L}^* -logic topology (see Definition 4.19) is profinite, and then so is L/K which is isomorphic to it.

The \mathcal{L} -logic topology on L/K is compact and Hausdorff and is weaker than the \mathcal{L}^* -logic topology which is also compact and Hausdorff. It follows that both topologies coincide. In particular L/K with the \mathcal{L} -logic topology is profinite so that $K = \bigcap H_i$ where H_i is \mathcal{L} -definable and $H_i \cap L$ is a subgroup of L . \square

6 Definable groups with f-generics in PRC

Proposition 6.1. *Let M be a model of T_{prc} and let $M_0 \prec M$, M is $|M_0|^+$ -saturated. Let G be a definable group in M and let p be a global type in G strongly f -generic over M_0 . Let $a \models p|_M$, $b \models p|_{Ma}$, and $c = ab$. Then $\text{tp}(c/Ma)$ is strongly f -generic over M_0 and there is an M -definable algebraic group H and dimension-generic elements $a', b', c' \in H(\mathcal{U})$ such that $a' \cdot b' = c'$ and $\text{acl}(Ma) = \text{acl}(Ma')$, $\text{acl}(Mb) = \text{acl}(Mb')$ and $\text{acl}(Mc) = \text{acl}(Mc')$.*

Proof. This is precisely Proposition 3.1 of [HP94]. We modified the statement to additionally require that we can choose a, b, c and the set A to be inside M , but the reader can verify that this conditions can be met by the construction done in the proof. \square

Definition 6.2. Let $(M, <_1, \dots, <_n)$ be a model of T_{prec} . We say that a definable set $X \subseteq M^m$ is *multi-semialgebraic* if X is a union of multi-cells in M^m . Let (G, \cdot_G) be an M -definable group. We say that G is *multi-semialgebraic* if G , the graph of \cdot_G and of the inversion of G are multi-semialgebraic.

Theorem 6.3. *Let $M \models T_{prec}$ be ω -saturated. Let G be an M -definable group with strong f -generics. Then there is a finite index M -definable subgroup $G_1 \leq G$, a finite $K \leq G_1$ central in G_1 , a multi-semialgebraic group H defined over M such that G_1/K is definably isomorphic to a finite index subgroup of $H(M)$.*

Proof. Let μ_M be the ideal of formulas which do not extend to a strongly bi- f -generic type over M . So μ_M is M -invariant, S1 and invariant under both left and right translations by elements of G . Let $q \in S(M)$ be μ_M -wide. By Theorem 3.18, $Stab(q) = G_M^{00}$ and μ_M -almost all elements of G_M^{00} are in $St(q)$.

Let $a \in G_M^{00}$ be such that $tp(a/M)$ is μ_M -wide. Let $b \models q$ such that $tp(b/Ma)$ is μ_M -wide and $tp(ab/M) = q$.

By Proposition 6.1 there is an M -definable algebraic group (H, \cdot_H) and $a', b', c' \in H$ such that $c' = a' \cdot_H b'$, $\text{acl}(Ma) = \text{acl}(Ma')$, $\text{acl}(Mb) = \text{acl}(Mb')$ and $\text{acl}(Mc) = \text{acl}(Mc')$.

We define an ideal μ on $G \times H$, by saying that $D \in \mu$ if and only if $\pi_1(D) \in \mu_M$. Then μ is M -invariant and invariant under left and right translations. We will refer to μ -wide as “wide”.

We define the ideal λ (that will define “medium” in Section 2) as the set of subsets X of $G \times H$ for which the projections to G and H each have finite fibers. Note that μ is S1 on medium types. Define $\tilde{p} = tp(a, a'/M)$. Then \tilde{p} is wide and medium. We will show that Theorem 2.13 can be applied to the type \tilde{p} .

Claim. Condition (A) holds: If p, q are two types in $G \times H$ and we have $(g, h) \models p \times_{nf} q$ such that either $tp(gh/M)$ or $tp(hg/M)$ is medium, then p is medium.

Proof. Denote $g = (g_0, g_1)$ and same for h . We will prove the case where we assume that $\text{tp}(gh/M)$ is medium, the other case is proved in an analogous way. Since $g_0h_0 \in \text{acl}(Mg_1h_1)$ we have $g_0 \in \text{acl}(Mg_1h_0h_1)$. As $\text{tp}(h_0h_1/Mg_0g_1)$ does not fork over M , this implies that $g_0 \in \text{acl}(Mg_1)$. In the same way we get $g_1 \in \text{acl}(Mg_0)$. \square

By Lemma 2.14, condition (B) holds. As T_{prec} is NTP_2 , condition (F) holds. We can then apply Theorem 2.13, which gives us a connected, medium, wide type-definable group $K \leq G \times H$. As K is medium, its projections to G and H have finite fibers. As K is wide, $\pi_1(K)$ is μ_M -wide and hence by connectedness $\pi_1(K) = G_M^{\text{oo}}$.

Claim. We may assume that π_1 and π_2 are injective on K .

Proof. For $i = 1, 2$, let $K_i = \pi_i^{-1}(e) \cap K$. Then K_i is finite and normal in K . As K is connected, K_i is central in K (the centralizer of K_1 is a relatively definable subgroup of K of finite index). Let $C \leq H$ be the centralizer of $\pi_2(K_1)$ inside H . It is an algebraic subgroup of H . Then we can replace H by $C/\pi_2(K_1)$ which is again an algebraic group (defined over the same parameters as H and K_1). Thus we may assume that K_1 is trivial. In the same way, replacing G by a subgroup of finite index, we may assume that $\pi_1(K_2)$ is central in G and then replace G by its quotient by $\pi_1(K_2)$. \square

Now choose a symmetric definable X_0 such that $K \subseteq X_0 \subseteq G \times H$, and such that π_1 and π_2 are injective on X_0^4 . Replacing G by a subgroup of finite index, we can assume that $\pi_1(X_0)$ generates G . By Proposition 2.11, K is normal in the group generated by X_0 and X_0^n is medium for any n . Observe that $\pi_1(X_0) \subseteq G$ is definably isomorphic to $\pi_2(X_0) \subseteq H$. In $\pi_2(X_0)$, the multi-topology τ is definable and the operations in H are continuous.

Since working with projections becomes quite messy, we will abuse notation in the following way:

- Any element of X_0 will be written as (x^*, x) where $x^* \in G$ and $x \in H$. So $x^* = \pi_1(\pi_2^{-1}(x))$ for any element x in $\pi_2(X_0) \subseteq H$. We will also do this for sets, so that $A^* = \pi_1(\pi_2^{-1}(A))$ for any $A \subset \pi_2(X_0)$.
- All non $*$ -elements will be assumed to belong to H . We will use Greek letters for elements in G which may not be in $\pi_1(X_0)$.
- We will identify K with $\pi_2(K)$.

- We will mostly be working inside H , so we will drop the index in \cdot_H .

In H we have that $\pi_2(X_0) \cap \overline{K}$ is generic in \overline{K} and by Proposition 5.3, \overline{K}/K is profinite, so there is a \mathcal{L} -definable set $X \subseteq \pi_2(X_0)$ such that $K_1 := X \cap \overline{K}$ is a subgroup of finite index of \overline{K} , and K_1^* is a type-definable subgroup of bounded index of G ($G_M^{00} \subseteq K_1^*$). By passing to a finite index subgroup of G , we may assume that X^* generates G .

Claim. We may assume that K_1 is a normal subgroup of \overline{K} , in fact normalized by X , and that K_1^* is a normal subgroup of G .

Proof. Let r be the smallest integer such that every $\gamma \in G$ is the \cdot_G -product of r elements in X . Define Y_1, \dots, Y_r such that:

$$Y_1 := \bigcap_{\gamma \in X} \gamma X \gamma^{-1},$$

$$Y_l := \bigcap_{\gamma \in X} \gamma Y_{l-1} \gamma^{-1}, \text{ for } 2 \leq l \leq r.$$

Then $(Y_r)^*$ is normalized by G , hence Y_r is normalized by X and so is $Y_r \cap \overline{K}$. We can now replace X by Y_r and K_1 by $Y_r \cap \overline{K}$. \square

Now, \overline{K} is the intersection of multi-semialgebraic sets in H . We can define a decreasing sequence $(U_k : k < \omega)$ of quantifier-free definable symmetric sets, such that:

- $U_0 = \overline{K}$;
- $(U_{m+1} \cap X)^3 \subseteq U_m \cap X$, for each $m < \omega$;
- $g(U_{m+1} \cap X)g^{-1} \subseteq U_m \cap X$, for each $m < \omega$ and $g \in X$;
- $\overline{K} = \bigcap_{k \in \omega} U_k$.

Note that by density of X in U_0 and by continuity of the operations, we also have $(U_{m+1})^3 \subseteq U_m$ and $gU_{m+1}g^{-1} \subseteq U_m$ for all $g \in X$.

Claim. We may assume that U_m are multi-open, for $m \geq 1$.

Proof. The type definable group \overline{K} has non empty interior in H (since it has bounded index). The operations are continuous in H and by definition X is dense in U_0 . It follows that, since $U_{m+1} \cdot \overline{K} \subseteq U_m$, every point in U_{m+1} has a neighborhood contained in U_m , so U_{m+1} is entirely contained in the interior of U_m . Replacing each U_m by its interior, we preserve the properties and the claim holds. \square

Select points $\{\alpha_k : k < p\}$ in G such that

$$G = \bigcup_{k < p} \alpha_k \cdot_G (U_4 \cap X)^*.$$

Note that for any $x \in G$, there is $k < p$ such that $x \in \alpha_k \cdot_G (U_4 \cap X)^*$ and then $x \cdot_G (U_4 \cap X)^* \subseteq \alpha_k \cdot_G (U_3 \cap X)^*$.

Let m be the smallest integer such that every α_i is the \cdot_G -product of m elements in $(U_3 \cap X)^*$.

Claim. For each i , the conjugation map $f_i : x \mapsto \pi_2(\pi_1^{-1}(\alpha_i \cdot_G x^* \cdot_G (\alpha_i)^{-1}))$ is an algebraic map from $U_{k+m} \cap X$ to $U_k \cap X$ for $k \geq 3$.

Proof. Let $\alpha_i = d_1^* \cdot_G \cdots \cdot_G d_m^*$, with each d_l in X . Then for any l , for any $j \geq 3$ and $x \in U_j \cap X$ the map $x \mapsto d_l^{-1} x d_l$ is algebraic (as H is algebraic) and by hypothesis $d_l^{-1} x d_l \in U_{j-1} \cap X$. Since

$$\pi_2(\pi_1^{-1}(\alpha_i x^* (\alpha_i)^{-1})) = d_l \cdots \cdots d_1 \cdot x \cdot d_1^{-1} \cdots \cdots d_l^{-1},$$

the function f_i is algebraic as a composition of algebraic functions. \square

Select points $\{b_i : i < l\}$ in $U_3 \cap X$ such that

$$U_3 \cap X = \bigcup_{i < l} b_i \cdot (U_{m+3} \cap X).$$

For $j < p$ and $r < l$, define $\alpha_{(j,r)} \in \{\alpha_k : k < p\}$ and $t_{(j,r)} \in (U_3 \cap X)$ such that (where all the products are in G):

$$\alpha_j^{-1} b_r^* \alpha_j = \alpha_{(j,r)} t_{(j,r)}^*.$$

Let $W = (U_3 \cap X) \times \{0, \dots, p-1\}$ and for $k < p$, define $W_k = (U_3 \cap X) \times \{k\}$.

Define an equivalence relation E on W^2 by $(x, i) E (y, j)$ if $\alpha_i \cdot_G x^* = \alpha_j \cdot_G y^*$. We then have

$$(x, i) E (y, j) \iff (y^*) \cdot_G (x^*)^{-1} = \alpha_j^{-1} \cdot_G \alpha_i.$$

If this happens, then $\alpha_j^{-1} \cdot_G \alpha_i$ lies in $(U_2 \cap X)^*$ and can be written as w_{ij}^* for some $w_{ij} \in U_2 \cap X$. When this is not the case, say that $w_{i,j}$ is undefined.

Note that we have a definably bijection $\phi : W/E \rightarrow G$ sending (x, i) to $\alpha_i \cdot_G x^*$.

We will now define a multi-semialgebraic group, which in a way will be the τ -topological closure of W/E .

Let $W^{cl} = U_3 \times \{0, \dots, p-1\}$ and $W_k^{cl} = U_3 \times \{k\}$. We equip each W_k^{cl} with the τ -topology. Then W_k is dense in W_k^{cl} .

We now define a relation E^{cl} on W^{cl} as follows: given $(x, i), (y, j) \in W^{cl}$, we have $(x, i)E^{cl}(y, j)$ if and only if w_{ij} is defined and $yx^{-1} = w_{ij}$.

Claim. E^{cl} is an equivalence relation.

Proof. Reflexivity holds as $w_{ii} = e$ for all i . Whenever w_{ij} is defined, then so is w_{ji} and $w_{ji} = w_{ij}^{-1}$. This implies symmetry. Finally, assume that $(x, i)E^{cl}(y, j)$ and $(y, j)E^{cl}(z, k)$, then $zx^{-1} = w_{jk}w_{ij} \in U_2 \cap X$ (as $zx^{-1} \in U_2$ and $w_{jk}w_{ij} \in X$). Then w_{ik} is defined and equal to $w_{jk}w_{ij}$ and thus $(x, i)E^{cl}(z, k)$. \square

By construction W/E embeds in W^{cl}/E^{cl} . We now define a group structure on W^{cl}/E^{cl} . First consider $(x, i), (y, j), (z, k) \in W$ and write $x = b_r w$ with $w \in U_{m+3} \cap X$. We then have, where all the products are understood in G :

$$\begin{aligned} & \alpha_i x^* \alpha_j y^* = \alpha_k z^* \\ \iff & \alpha_i b_r^* w^* \alpha_j y^* = \alpha_k z^* \\ \iff & \alpha_i \alpha_j \alpha_{(j,r)} t_{(j,r)}^* f_j(w)^* y^* = \alpha_k z^* \\ \iff & t_{(j,r)}^* f_j(w)^* y^* (z^*)^{-1} = \alpha_{(j,r)}^{-1} \alpha_j^{-1} \alpha_i^{-1} \alpha_k. \end{aligned}$$

When such an equation holds, we define $\epsilon(i, j, k, r)$ as $\alpha_{(j,r)}^{-1} \alpha_j^{-1} \alpha_i^{-1} \alpha_k \in U_1 \cap X$. Let $\Gamma \in W^3$ be the pullback of the graph of multiplication on $W/E \cong G$ via the canonical projection. Then $((x, i), (y, j), (z, k)) \in \Gamma$ if and only if $\epsilon(i, j, k, r)$ is defined and writing $x = b_r w$, we have:

$$t_{(j,r)} f_j(w) y z^{-1} = \epsilon(i, j, k, r). \quad (E\Gamma)$$

We define Γ^{cl} on W^{cl} by $((x = b_r w, i), (y, j), (z, k)) \in \Gamma^{cl}$ if $(E\Gamma)$ holds. We need to check that this is well defined, *i.e.*, does not depend on the decomposition of x as $b_r w$. So assume that $x = b_r w = b_s w'$. Then $w' = b_s^{-1} b_r w$. Assume that $t_{(j,r)} f_j(w) y z^{-1} = \epsilon(i, j, k, r)$. On a small neighborhood of (w, y, z)

we can find (w_0, y_0, z_0) , all points lying in X such that $t_{(j,r)}f_j(w_0)y_0z_0^{-1} = \epsilon(i, j, k, r)$ (as all operations are continuous). Set $w'_0 = b_s^{-1}b_rw_0$, then w'_0 is close to w' , hence in $U_{m+3} \cap X$ and we have $t_{(j,s)}f_j(w'_0)y_0z_0^{-1} = \epsilon(i, j, k, s)$ (in particular $\epsilon(i, j, k, s)$ is defined). Letting (w_0, y_0, z_0) converge to (w, y, z) , we obtain $t_{(j,s)}f_j(w')yz^{-1} = \epsilon(i, j, k, s)$ as required.

A similar argument shows that Γ^{cl} is E^{cl} -equivariant: if say $(z, k)E^{cl}(z', k')$, then we have $z' = w_{ii'}z$ and we conclude as above that $((x, i), (y, j), (z, k))$ is in Γ^{cl} if and only if $((x, i), (y, j), (z', k'))$ is in Γ^{cl} . Therefore Γ^{cl} induces a ternary relation on the quotient W^{cl}/E^{cl} . Note that on each $W_i^{cl} \times W_j^{cl} \times W_k^{cl}$, Γ^{cl} is the closure of Γ .

Claim. Γ^{cl} induces the graph of a function $W^{cl}/E^{cl} \times W^{cl}/E^{cl} \rightarrow W^{cl}/E^{cl}$.

Proof. First, assume that $(x, i), (y, j) \in W^{cl}$, $x = b_rw$. Then for a given j , the equation $t_{(j,r)}f_j(w)yz^{-1} = \epsilon(i, j, k, r)$ can have at most one solution in z . If we can find (z', k') such that $t_{(j,r)}f_j(w)yz'^{-1} = \epsilon(i, j, k', r)$ also holds, then $\epsilon(i, j, k', r)w_{kk'} = \epsilon(i, j, k, r)$ and so $z'^{-1}w_{kk'} = z^{-1}$ which implies $(z, k)E^{cl}(z', k')$. This shows that the image is unique.

It remains to show existence. Take $(x, i), (y, j) \in W^{cl}$. Take a small neighborhood U_* of the identity included in \overline{K} . Then there are some k and r such that for any $x_0 \in xU_*$ and $y_0 \in yU_*$, there is (z_0, k) with $((x_0, i), (y_0, j), (z_0, k)) \in \Gamma$ and x_0 can be written as b_rw_0 with $w_0 \in U_{m+4} \cap X$. We may also assume that for any such z_0 , $z_0\overline{K} \subseteq U_3$. Then we have $t_{(j,r)}f_j(w_0)y_0z_0^{-1} = \epsilon(i, j, k, r)$. We can then write $x = b_rw$ for some $w \in U_{m+3} \cap X$ and define $z = \epsilon(i, j, k, r)^{-1}t_{(j,r)}f_j(w)y$. Then $z \in z_0\overline{K} \subseteq U_3$ and $((x, i), (y, j), (z, k)) \in \Gamma$. \square

Let \odot the boolean function induced by Γ on W^{cl}/E^{cl} . As associativity is a closed condition Γ^{cl} is the closure on Γ on each $W_i \times W_j \times W_k$, \odot is associative. Existence of inverses is proved as the existence part of the previous claim, fixing $z = e$ and looking for y . Therefore we have equipped W^{cl}/E^{cl} with a group structure. Write this group as G_0 .

The sets W_k^{cl} are multi semialgebraic and E^{cl} and Γ^{cl} are algebraic. Strictly speaking, G_0 thus constructed is not multi semialgebraic as its construction involves quotients. However one can easily remove the quotients: letting $\pi : W \rightarrow G_0$ be the quotient map, G_0 is in definable bijection with a multi semialgebraic group with underlying set

$$W_0 \cup (W_1 \setminus \pi^{-1}(\pi(W_0))) \cup (W_2 \setminus \pi^{-1}(\pi(W_1) \cap \pi(W_0))) \cup \dots$$

As G embeds definably into G_0 as a subgroup of finite index. This finishes the proof of the theorem. \square

6.1 Additional comments

- As already pointed out in the introduction, we in fact obtain a stronger statement than stated in Theorem 6.3: A finite index subgroup of G is isogeneous with a finite index subgroup of a ‘multi-Lie group’ that is a group which admits a definable manifold structure, in the sense of the multi-topology, for which the group operations are continuous (even C^∞ with respect to each order).
- Will Johnson has studied in [Joh13] the model companion of fields with n distinct orderings. This is a particular case of bounded PRC fields. Johnson proves that a Lascar-invariant quantifier-free type extends to a Lascar-invariant measure. It seems likely that an adaptation of those results should show that in this case, any group with f-generics has a translation-invariant measure.
- We expect those results to generalize to groups definable in the main sort of a pseudo p-adically closed field. This will be dealt with in future work.

Appendix: Shelah expansion and NTP_2

Theorem 6.4. *Let T be NTP_2 in a language L and assume that we have an expansion T' of T to a language L' by externally definable sets. Assume furthermore that T' has elimination of quantifiers in L' and the only additional predicates in L' are traces of externally definable NIP formulas. Then T' is NTP_2 .*

Proof. Let $M \models T'$ be \aleph_1 -saturated and let $M \prec N$ be $|M|^+$ -saturated. The property of NTP_2 for formulas is preserved by finite disjunctions, but not by finite conjunctions in general. It is enough to show that a formula of the form $\phi(x; y) \wedge \psi(x; y)$ is NTP_2 , where $\phi(x; y) \in L$ and $\psi(x; y) \in L'$ is such that there is an NIP L -formula $\tilde{\psi}(x; y; d) \in L(N)$ such that $\psi(M) = \tilde{\psi}(N; d) \cap M$. Let (N, M) denote the expansion of N with a new unary predicate naming M . Let $(N_1, M_1) \succ (N, M)$ be a sufficiently saturated elementary extension.

By honest definitions, there is $\theta(x, y; e) \in L(M_1)$ such that $\theta(M; e) = \psi(M)$ and $\theta(M_1; e) \subseteq \tilde{\psi}(M_1; d)$. Note that the formula $\theta(x, y; e)$ could have IP.

Now assume that we are given a witness of TP_2 for $\phi(x; y) \wedge \psi(x; y)$. Namely, we have an array $(b_{i,j} : i, j < \omega)$ and some k such that each line $\{\phi(x; b_{i,j_0}) \wedge \psi(x; b_{i,j_0}) : i < \omega\}$ is k -inconsistent and for every $\eta : \omega \rightarrow \omega$, the path $\{\phi(x; b_{\eta(j),j}) \wedge \psi(x; b_{\eta(j),j}) : j < \omega\}$ is consistent, hence realized by some $a_\eta \in M$. Now the properties of the array are preserved if we replace the formula $\phi(x; y) \wedge \psi(x; y)$ by $(\phi(x, y) \wedge \theta(x, y; e))$: the paths are still consistent, using the same witnesses a_η , and the lines are still k -inconsistent (in the structure M_1) by the honesty property. This shows that the formula $\phi(x, y) \wedge \theta(x, y; e)$ has TP_2 in M_1 which contradicts the hypothesis that T is NTP_2 . \square

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